Visual Tracking via Geometric Particle Filtering on the Affine Group with Optimal Importance Functions

Junghyun Kwon¹, Kyoung Mu Lee¹, and Frank C. Park²

¹Department of EECS, ²School of MAE
Seoul National University, Korea
Affine Motion Tracking

Estimation of $p(X_k \mid y_{1:k})$ via \textit{particle filtering} (PF)

- Affine motion tracking $\Rightarrow$ \textit{Non-linear filtering}

Particle filtering $\Rightarrow$ Effective for non-linear filtering
Issues in Affine Motion Tracking via PF

- Adequate state representation

State space $\rightarrow$ A set of 2-D affine matrices

$$\begin{bmatrix} G & t \\ 0 & 1 \end{bmatrix}$$

- Use of optimal importance function (OIF)

PF $\rightarrow$ Sampling from the importance function

OIF is essential for robust tracking
Two Different State Representations

<table>
<thead>
<tr>
<th>State</th>
<th>Conventional</th>
<th>Geometric (Kwon et al, 2008)</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>6-D vector by local coordinates</td>
<td>2-D Affine matrix itself as a Lie group (Aff(2))</td>
</tr>
<tr>
<td>State Equation</td>
<td>$x_k = f(x_{k-1}) + w_k$</td>
<td>$X_k = X_{k-1} \cdot \exp \left( A(X,t) \Delta t + dW_k \sqrt{\Delta t} \right)$</td>
</tr>
</tbody>
</table>

Ignores geometry of the underlying space  

We take this Geometric approach!
Drawback of Conventional Approach

Toy example

\[ X_{k-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ x_{k-1} = (1, 0, 0, 0, 1, 0)^T \]

\[ X'_{k-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ x'_{k-1} = (2, 0, 0, 0, 2, 0)^T \]

\[ x_k = x_{k-1} + w_k \]

\[ x'_k = x'_{k-1} + w_k \]

State farther from I \(\rightarrow\) Need Larger perturbation \(\rightarrow\) Need more particles \(\rightarrow\) Inefficiency
For the same example

$$X_{k-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X_{k-1}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X_k = X_{k-1} \cdot \exp(dW_k)$$

$$X_k' = X_{k-1}' \cdot \exp(dW_k)$$

More efficient than the conventional approach

**Affine motion tracking via Geometric PF on Aff(2)!**
Remaining Issue: Use of OIF

Popular importance function $\Rightarrow p(X_k | X_{k-1})$

Direct sampling from the state equation

Simple but inefficient because of missing $y_k$

Optimal importance function $\Rightarrow p(X_k | X_{k-1}, y_k)$

Maintaining the largest number of effective particles

Increased effective particles $\Rightarrow$ Increased performance
How to Use OIF in Practice

Difficulties in using OIF for Affine Motion Tracking

1. Only approximation to OIF is possible

2. Our state \( \Rightarrow 2\)-D affine group \( \text{Aff}(2) \)

Our contribution

- Approximation of OIF for \textit{geometric PF on Aff}(2)
- Consideration of the \textit{geometry of Aff}(2)
OIF Approximation for Vector State

- **Gaussian approximation** by (Doucet et al, SC2000)

Predicting $p(x_k, y_k | x_{k-1})$ by Jacobian of $y_k$ w.r.t. $x_k$

$p(x_k | x_{k-1}, y_k)$ by correcting using $y_k$
Questions in OIF Approx. on Aff(2)

Q1. What is Gaussian on Aff(2)?

Q2. How to obtain Jacobian of $y_k$ w.r.t. $X_k$?

Our answer

Use of exponential coordinates!
Geometric View to Affine Matrix

Affine matrix $\rightarrow$ 2-D Affine group $\text{Aff}(2)$

Lie group $\rightarrow$ Group + Differentiable manifold

Lie algebra $\rightarrow$ Tangent space at the identity ($\text{aff}(2)$)

Exp: $\text{aff}(2) \rightarrow \text{Aff}(2)$
Log: $\text{Aff}(2) \rightarrow \text{aff}(2)$
Local Diffeomorphism of Exp Map

Local diffeomorphism of Exp: aff(2) → Aff(2)

One-to-one and onto sufficiently near the identity
Exponential Coordinates

- By **local diffeomorphism** of Exp map

\[ X = \exp \left( \sum_{i=1}^{6} u_i E_i \right) \rightarrow (u_1, \ldots, u_6)^T \]

- Neighborhood of \( \text{Aff}(2) \)

\[ Y(u) = Y \cdot \exp \left( \sum_{i=1}^{6} u_i E_i \right) \]
A1. Gaussian on Aff(2)

Motivation ➔ Gaussian is *well defined on* \( \text{aff}(2) \)!

\[ N_{\text{Aff}(2)}(X, S) \Rightarrow \text{Exponential of Gaussian on } \text{aff}(2) \]

\[
X \cdot \exp \left( \sum_{i=1}^{6} \varepsilon_i E_i \right), \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_6)^T \sim N(0, S)
\]

Constraint ➔ *Sufficiently small* \( S \)
A2. Measurement Jacobian

- Taylor expansion on $\text{Aff}(2)$ by *Exp coordinates*

$$y_k = g(X_k) + n_k$$
$$\approx g(X_k) + J \cdot u + n_k$$

$$J_i = \left. \frac{\partial g(X_k(u))}{\partial u_i} \right|_{u=0}$$

*\( J \) with respect to the *exponential coordinates*

*\( J \) of *PCA-based measurement* by the *chain rule*

$$y_k = h(I(w(p; X_k))) + n_k = \left( \sum_{i=1}^{M} \frac{c_i^2}{\lambda_i} \right) + n_k$$
OIF Approximation on Aff(2)

\[ X_k = X_{k-1} \cdot \exp \left( A(X, t) \Delta t + dW_k \sqrt{\Delta t} \right) \]

\[ f(X_{k-1}) = X_{k-1} \cdot \exp (A(X, t) \Delta t) \]

**Prediction by \( J \)**

\[ \text{Exp} \]

\[ p(X_k \mid X_{k-1}) \approx N_{\text{Aff}(2)} \left( f(X_{k-1}), P \Delta t \right) \]

**Correction**

\[ \text{Exp} \]

\[ p(X_k \mid X_{k-1}, y_k) \approx N_{\text{Aff}(2)} \left( m_k, \Sigma_k \right) \]
Experiment 1

Geometric approach vs Conventional approach

Our tracker vs Tracker of (Ross et al, IJCV 2008)

Same measurement, Same importance function

3x speed playback  2x speed playback
Experiment 2

🔥 Approx. OIF vs State prediction density

Our tracker with Approx. OIF vs with \( p(X_k | X_{k-1}) \)

Same # of particles, Same covariance for \( dW_k \)
## Experiment 2

**Approx. OIF vs State prediction density**

<table>
<thead>
<tr>
<th>Importance function</th>
<th>Cube</th>
<th>Vase</th>
<th>Toy</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Approx. OIF</strong></td>
<td>43.43</td>
<td>13.69</td>
<td>15.69</td>
</tr>
<tr>
<td>$p(X_k</td>
<td>X_{k-1})$</td>
<td>18.12</td>
<td>8.37</td>
</tr>
</tbody>
</table>

### Number of effective particles

$\left[ \sum (w^{(i)}_k)^2 \right]^{-1}$
Conclusions

- **Geometric framework** to approx. OIF for PF on Aff(2)

  - Use of Exponential coordinates
  - Approx. Gaussian and Taylor expansion on Aff(2)

- **Experimental validation**

  - Efficiency of geometric PF on Aff(2)
  - Efficiency of OIF for geometric PF on Aff(2)
Thanks for your attention!

http://cv.snu.ac.kr
Additional Slides
Conventional Approach

6-D Vector representation using local coordinates

Via Euclidean Embedding

\[ X = \begin{bmatrix} G & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow x = (a_1, \ldots, a_6)^T \in \mathbb{R}^6 \]

Via Singular Value Decomposition

\[ G = \text{Rot}(\phi_1)\text{Rot}(-\phi_2)\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}\text{Rot}(\phi_2) \]

\[ \rightarrow x = (\phi_1, \phi_2, s_1, s_1/s_2, a_3, a_6)^T \in \mathbb{R}^6 \]
Physical Illustration

$X_{k-1}$  $X_k$

$X'_{k-1}$  $X'_k$

$X_0$  $X'_0$

Camera
**Explicit** representation to apply the chain rule

\[ y_k = h(I(w(p; X_k))) + n_k = \left( \sum_{i=1}^{M} \frac{c_i^2}{\lambda_i} \right) + n_k \]

Initial coordinates

- **PCA-based Measurement**

- \( X_k = \begin{bmatrix} G & t \\ 0 & 1 \end{bmatrix} \)

- **DFFS**

- **DIFS**

- **k-th frame**
Application of the Chain Rule

**Straight-forward** application and calculation

Derivative of DFFS and DIFS

\[ J_i = \left. \frac{\partial g(X(u))}{\partial u_i} \right|_{u=0} = \frac{\partial h}{\partial I} \cdot \frac{\partial I}{\partial w} \cdot \frac{\partial w}{\partial X(u)} \cdot \left. \frac{\partial X(u)}{\partial u_i} \right|_{u=0} \]

Derivative of warping function

Image gradient

\[ X_k E_i \]

Now OIF approximation for PF on Aff(2) possible!
Derivative of PCA Term

Derivative of DFFS term

\[
e^2 = \sum_p (I(p) - \overline{T}(p))^2 - \sum_{i=1}^M c_i^2
\]

\[
\frac{\partial e^2}{\partial I(p)} = \sum_p \left(2(I(p) - \overline{T}(p)) - \sum_{i=1}^M 2c_i b_i(p)\right)
\]

Derivative of DIFS term

\[
\frac{\partial}{\partial I(p)} \frac{\sum_{i=1}^M c_i^2}{\lambda_i} = \sum_p \left(\sum_{i=1}^M \frac{2c_i}{\lambda_i} b_i(p)\right)
\]
Approx. OIF for PF on Aff(2)

**Prediction and Correction**

| Step 1 | Prediction of \( p(X_k, y_k | X_{k-1}) \) by \( J \) |
|---|---|
| \( \mu_1 = f(X_{k-1}) \) | \( \Sigma_{11} = Q = P \Delta t, \Sigma_{12} = QJ^T \) |
| \( \mu_2 = g(f(X_{k-1})) \) | \( \Sigma_{22} = JQJ^T + R \) |

| Step 2 | \( p(X_k | X_{k-1}, y_k) \) by correction |
|---|---|
| \( \bar{u} = \Sigma_{12} \Sigma_{22}^{-1} (y_k - \mu_2) \) |
| \( N_{Aff(2)}(m_k, \Sigma_k) \) | \( m_k = \mu_1 \cdot \exp \left( \sum_{i=1}^{6} u_i E_i \right) \) |
| \( \Sigma_k = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^T \) |